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Berry phases and Hamiltonian time dependence

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Abstract. We show that (non-adiabatic) Berry phases in time-independent systems arise for fundamentally different reasons than those for time-dependent systems. For time-independent Hamiltonians the Berry phases are seen to only depend on the chosen cyclic initial state, the Hamiltonian merely providing the appropriate period. We also discuss two time-dependent examples where the time dependence arises from different sources, namely a two-level atom in an intense laser and a harmonic oscillator with a periodic forcing term.

1. Introduction

In this paper we discuss the relationship between the Berry phase and the time dependence of the Hamiltonian. In section 2 we show that the origin of Berry phases for time-independent systems differs fundamentally from their origin in time-dependent systems and prove that for constant Hamiltonians the details of the Hamiltonian are only relevant in finding the appropriate period, the numerical values of the Berry phases being Hamiltonian independent. In section 3 we discuss how time dependence can arise in quantum systems.

We then discuss some examples of the Berry phase for time-dependent systems. In section 4 we discuss the Berry phases for two-level atoms in strong near-resonant fields. In this system the time dependence is generated by tracing out the experimentally irrelevant photon states in the density operator and showing that the reduced electronic density operator is generated by a time dependent Hamiltonian. This gives the same result as the well known heuristic method of replacing the creation operator for the boson field by the function $e^{i\omega t}$. In section 5 we discuss another system involving bosons, the forced harmonic oscillator. Here we find that the existence or otherwise of any cyclic initial states is dependent on the non-existence or otherwise of a single Fourier component in the periodic forcing term. For convenience we set $\hbar$ to be unity.

2. Time-independent systems

Consider a constant Hamiltonian H and some arbitrary time $\tilde{t}$. As H may be regarded as being $\tilde{t}$-periodic we may use the Floquet theory formalism of Moore and Stedman (1990b), decomposing the evolution operator U into the product form $U = Ze^{iMt}$

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where Z is unitary and $\tilde{t}$-periodic and M is self-adjoint and constant. We find that the eigenvectors $\phi_\alpha(0)$ of M are cyclic initial states and that they have Berry phases

$$\gamma_\alpha = \mathrm{i}\int_0^{\tilde{t}} \langle \phi_\alpha(0)|Z^*\dot{Z}|\phi_\alpha(0)\rangle \,\mathrm{d}t. \tag{1}$$

Now a particular choice for Z and M is not unique as given one choice an equally valid one is

$$Z' = Z\mathrm{e}^{-2\pi\mathrm{i}Nt/\tilde{t}} \tag{2}$$

$$M' = M + 2\pi N/\tilde{t} \tag{3}$$

where N is an arbitrary operator whose eigenvectors are the cyclic initial states of H and all of whose eigenvalues are integral. This does not usually cause any problem, as in most cases all possible choices are equally valid. However there is one important exception; if one choice of $M\tilde{t}$ is quasi-degenerate in the sense that two of its eigenvalues differ by an integral multiple of 2π then there exists an $M'\tilde{t}$ that has truly degenerate eigenvalues. If this is the case there will exist eigenvectors $\phi(0)$ of M' that are not eigenvectors of M so that not all of the cyclic initial states will be eigenvectors of M. We will show that for a constant Hamiltonian we can always choose a period $\tilde{t}$ such that two given cyclic initial states become quasi-degenerate and that it is this quasi-degeneracy that allows us to get non-zero Berry phases for time independent systems. This can be compared to time-dependent systems which can have non-zero Berry phases in the abscence of quasi-degeneracy.

For our constant Hamiltonian H we may choose $Z = I$ and $M = -H$. If $\tilde{t}$ is such that no two eigenstates of $M\tilde{t}$ are quasi-degenerate then this choice is adequate for a complete description of the cyclic initial states. Thus as $Z^*\dot{Z} = 0$ all of the Berry phases must vanish. However given any two eigenstates ϕ_α and ϕ_β of H with energies E_α and E_β respectively, taking $E_\beta > E_\alpha$ for convenience, we can choose

$$\tilde{t} = 2\pi(E_\beta - E_\alpha)^{-1} \tag{4}$$

making ϕ_α and ϕ_β quasi-degenerate so that any initial state of the form

$$\psi(0) = a_\alpha\phi_\alpha + a_\beta\phi_\beta \tag{5}$$

is cyclic. This quasi-degeneracy can be transformed into a real degeneracy by transforming Z and M as in (2) and (3) with the diagonal operator N with eigenvalues $n_\alpha = 0$ and $n_\beta = 1$. This gives

$$Z'^*\dot{Z}' = -2\pi\mathrm{i}N/\tilde{t} \tag{6}$$

so that

$$\mathrm{i}\int_0^{\tilde{t}} Z'^*\dot{Z}'\mathrm{d}t = 2\pi N. \tag{7}$$

Thus from (1) the cyclic initial state $a_\alpha\phi_\alpha + a_\beta\phi_\beta$ has Berry phase

$$\gamma = 2\pi|a_\beta|^2. \tag{8}$$

We may use this general result to rederive the Berry phases for the Hamiltonian

$$H = -\mu B \sigma_z \tag{9}$$

discussed by Aharonov and Anandan (1987). This has eigenvectors

$$\phi_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

$$\phi_\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

with energies $E_\alpha = -\mu B$ and $E_\beta = \mu B$ respectively. Thus to get Berry phases we must choose $\tilde{t} = \pi/\mu B$ and so for the arbitrary cyclic initial state

$$\psi(0) = \begin{pmatrix} \cos \frac{1}{2}\theta \\ \sin \frac{1}{2}\theta \end{pmatrix} \tag{12}$$

we get the Berry phase from (8)

$$\gamma = \pi(1 - \cos\theta) \tag{13}$$

in agreement with the previous authors.

Thus we have seen that Berry phases for time-independent systems arise for a different reason than those for time-dependent systems. That is, they arise due to the fact that for a certain choice of period $\tilde{t}$ two or more cyclic initial states become degenerate eigenvectors of M. This means that an arbitrary linear combination of them is also a cyclic initial state and we find that it is these states that have non-zero Berry phases.

3. Time-dependent systems

We now turn to time-dependent systems, which can exhibit non-zero Berry phases when there is no quasi-degeneracy. As an isolated system must have a time-independent Hamiltonian, the time dependence, and so the Berry phases of the system, must come from an interaction of the system with its surroundings. Two examples of such an interaction have already been given.

In Berry's original paper on the adiabatic Berry phase (Berry 1984) the time dependence comes from external parameters that are varied adiabatically about some closed path. This is effectively a classical treatment of the surroundings. We note that if the variation of the parameters is non-adiabatic we still get a time-dependent Hamiltonian and we may find its non-adiabatic Berry phases. As Anandan (1988) has pointed out the variation does not even have to be cyclic; however for this case the Hamiltonian is not periodic and so we cannot use the Floquet formalism.

Other authors have looked at the situation where the external fields are themselves quantised by using the Born–Oppenheimer approximation (Moody *et al* 1986). In this approach one can either concentrate on the fast variables, in which case one gets a Berry phase by varying the slow variables over a closed path in the spirit of the classical treatment of the surroundings, or one can concentrate on the slow variables whereupon the Berry phase manifests itself as a gauge field in the effective slow Hamiltonian.

In the following we discuss two systems involving bosons in some way. In the first we discuss a third way of generating time dependence in our system. We start with a time-independent Hamiltonian involving coupling between bosons and fermions, in this case the interaction between light and a two-level atom. When we trace out the boson states in the joint density operator to get an electronic reduced density operator we find that the effective electronic Hamiltonian is time dependent. The second system we discuss is an example of Berry's method for generating time dependence, that is we have a boson Hamiltonian containing a time-dependent forcing term that arises due to some unspecified coupling of the classical surroundings with the quantum system. This system exhibits unusual behaviour in that the presence or otherwise of a certain Fourier component in the forcing term determines whether or not any cyclic initial states exist. If this component is not present we find that every state is cyclic, providing the ultimate example of quasi-degeneracy, whereas if the component is not present no state is cyclic.

The two-level atom example provides a system where the Berry phases in the electronic subsystem are directly interpretable in terms of the energy levels of the time-independent Hamiltonian of the joint electron–photon system and are readily experimentally verifiable. In fact we find that the overall phases for the two cyclic initial states of the electronic subsystem are directly related to the Rabi oscillation frequency and the splitting of the Mollow triplet, two well known quantities in quantum optics. This relationship has also been discussed by Tewari (1989) in the adiabatic context. We use the Floquet Hamiltonian approach of Moore (1990) and exploit the relationship between the fictitious Floquet Hamiltonian and the actual joint electron–photon Hamiltonian noted by Shirley (1965).

4. The two-level atom

Here we discuss the Berry phases for two-level atoms in strong near resonant fields. Consider an atom in a strong laser beam that is nearly in resonance with one of the gaps in the atomic spectrum, say between the two states $|+\rangle$ and $|-\rangle$. Then to a good approximation the system behaves as a two-level atom with joint electron–photon Hamiltonian

$$H = E\sigma_z + \omega b^* b + \lambda(\sigma_+ b + \sigma_- b^*) \tag{14}$$

in the product basis $\{|\pm, n\rangle\}$, where $|n\rangle$ is a photon number state and we have made use of the rotating wave approximation. To calculate the semiclassical electronic Hamiltonian we will derive the density operator ρ for the joint system, trace out the photon label n to find the electronic reduced density operator ρ_a and show that ρ_a is generated by a Hamiltonian H_a. This is the semiclassical Hamiltonian.

4.1. The semiclassical Hamiltonian

We assume that the density operator initially represents a pure state that factorizes, i.e. the system is initially in the pure state

$$\psi(0) = \psi_a(0) \otimes \psi_p(0) \tag{15}$$

where $\psi_a(0)$ is a general atomic state

$$\psi_a(0) = a_+|+\rangle + a_-|-\rangle \tag{16}$$

and $\psi_p(0)$ is the coherent state

$$\psi_p(0) = \mathrm{e}^{-|z|^2/2} \sum_{n=0}^{\infty} z^n (n!)^{-1/2} |n\rangle. \tag{17}$$

We choose the coherent state as this corresponds to the classical limit of the laser output (Loudon 1983). For inessential simplicity we take z to be real, corresponding to an initial phase of zero for the semiclassical photon field.

To calculate the evolved state $\psi(t)$ and so the density operator ρ, we decompose $\psi(0)$ in (15) into a sum of eigenstates of the Hamiltonian H in the standard way. Now it may readily be verified that H has eigenvectors

$$|\epsilon_g\rangle = |-, 0\rangle \tag{18}$$

$$|\epsilon_{\pm,n}\rangle = x_{\pm,n}|-, n+1\rangle \mp x_{\mp,n}|+, n\rangle \tag{19}$$

with eigenvalues

$$\epsilon_g = -E \tag{20}$$

$$\epsilon_{\pm,n} = (n + \tfrac{1}{2})\omega \mp \theta_n \tag{21}$$

where n is a non-negative integer, $\theta_n^2 = (n+1)\lambda^2 + (E - \omega/2)^2$ and $x_{\pm,n} = (2\theta_n)^{-1/2}(\theta_n \mp \omega/2 \pm E)^{1/2}$. Now we are assuming that the laser is strong so that z is large. This means that the Poisson distribution of photon number state occupancies is strongly peaked about the mean $\bar{n} = z^2$. Thus we can ignore the contribution to the initial state of terms with small n, in particular the $n = 0$ terms. With this simplification we find that

$$\psi(t) = \mathrm{e}^{-z^2/2} \sum_{n=1}^{\infty} z^n (n!)^{-1/2} \mathrm{e}^{-\mathrm{i}n\omega t}(c_{+,n}|+, n\rangle + \mathrm{e}^{\mathrm{i}\omega t} c_{-,n}|-, n\rangle) \tag{22}$$

where

$$c_{+,n} = a_+(x_{-,n}^2 p_{+,n} + x_{+,n}^2 p_{-,n}) + z(n+1)^{-1/2} a_- x_{+,n} x_{-,n}(p_{-,n} - p_{+,n}) \tag{23}$$

$$c_{-,n+1} = a_-(x_{+,n}^2 p_{+,n} + x_{-,n}^2 p_{-,n}) + z^{-1}(n+1)^{1/2} a_+ x_{+,n} x_{-,n}(p_{-,n} - p_{+,n}) \tag{24}$$

and $p_{\pm,n} = \exp(-\mathrm{i}t(\omega/2 \mp \theta_n))$.

Further, as the photon number state occupancy distribution is very narrow we can replace n in the expression for θ_n with its mean value z^2 giving $\theta^2 = k^2 + (E - \omega/2)^2$ where $k = z\lambda$ is the effective interaction parameter. Thus $x_{\pm,n}$, $p_{\pm,n}$ and $c_{\pm,n}$ can be taken to be independent of n so that, restoring the $n = 0$ term for convenience, (22) simplifies to

$$\psi(t) = \mathrm{e}^{-z^2/2} \sum_{n=0}^{\infty} z^n (n!)^{-1/2} \mathrm{e}^{-\mathrm{i}n\omega t}(c_+|+, n\rangle + \mathrm{e}^{\mathrm{i}\omega t} c_-|-, n\rangle). \tag{25}$$

Using the fact that the density operator is given by $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ we can easily trace out the photon label n to give the electronic reduced density operator

$$\rho_a(t) = c_+\bar{c}_+|+\rangle\langle+| + c_+\bar{c}_-\mathrm{e}^{-\mathrm{i}\omega t}|+\rangle\langle-| + c_-\bar{c}_+\mathrm{e}^{\mathrm{i}\omega t}|+\rangle\langle-| + c_-\bar{c}_-|-\rangle\langle-| \tag{26}$$

where

$$c_+\bar{c}_+ = |a_+|^2 + 2x_+^2x_-^2(|a_-|^2 - |a_+|^2)(1-\cos\mu) + a_+\bar{a}_-x_+x_-(x_+^2 - x_-^2 - x_+^2p_-\bar{p}_+ + x_-^2p_+\bar{p}_-) + a_-\bar{a}_+x_+x_-(x_+^2 - x_-^2 - x_+^2p_+\bar{p}_- + x_-^2p_-\bar{p}_+) \tag{27}$$

$$c_-\bar{c}_- = 1 - c_+\bar{c}_+ \tag{28}$$

$$c_+\bar{c}_- = (|a_+|^2 - |a_-|^2)x_+x_-(x_+^2 - x_-^2 - x_+^2p_-\bar{p}_+ + x_-^2p_+\bar{p}_-) + a_+\bar{a}_-(2x_+^2x_-^2 + x_+^4p_-\bar{p}_+ + x_-^4p_+\bar{p}_-) + 2a_-\bar{a}_+x_+^2x_-^2(1-\cos\mu) \tag{29}$$

and $\mu = -(\epsilon_{+,n} - \epsilon_{-,n})t$.

By direct substitution of (26) into the Liouville equation

$$i\dot{\rho}_a = [H_a, \rho_a] \tag{30}$$

one can readily show that the electronic reduced density operator ρ_a is generated by the electronic semiclassical Hamiltonian

$$H_a = \begin{bmatrix} E & ke^{-i\omega t} \\ ke^{i\omega t} & -E \end{bmatrix}. \tag{31}$$

Thus we can explicitly see how a semiclassical field induces time dependence in the Hamiltonian of the fermionic sytem upon which it acts. We note that the semiclassical limit for this problem corresponds to simple-mindedly replacing the boson creation operator b^* by the function $ze^{i\omega t}$.

4.2. Berry phases

The Berry phases for this Hamiltonian have already been calculated by Moore (1990). The cyclic initial states are given by

$$\phi_\pm(0) = \mp x_\mp|+\rangle + x_\pm|-\rangle \tag{32}$$

and have overall phases

$$\chi_\pm = (-\omega/2 \pm \theta)\tilde{t}. \tag{33}$$

where $\tilde{t} = 2\pi/\omega$ and $x_\pm$ and θ are as before. Thus we can see that $\mu = (\chi_+ - \chi_-)t/\tilde{t}$ meaning that information about the difference in overall phase for the two cyclic initial states is contained in the reduced density operator. This is because there is only one arbitrary phase in quantum mechanics. In the following we show that this phase difference is measurable and in fact has already been measured on numerous occasions. This has already been noted by Tewari (1989) in the context of adiabatic evolution. First we will show that the difference in overall phase is related to the well known Rabi oscillation frequency. This follows immediately from our expression for ρ_a. Choose $a_- = 1$ and $a_+ = 0$ so that the atom is initially in the ground state. Then the expectation value of the electronic energy operator, $H_e = E\sigma_z$ is given by

$$\langle H_e\rangle(t) = -E[1 - 4x_+^2x_-^2(1-\cos\mu)] \tag{34}$$

which reduces to the standard expression for the Rabi oscillation (Allen and Eberly 1975) when we substitute for $x_\pm$.

We will now show that the difference in overall phases is related to the splitting of the Mollow triplet. This is most easily seen by exploiting the relationship between the fictitious Floquet Hamiltonian H_F, used here to calculate the Berry phases, and the joint electron–photon Hamiltonian H first noted by Shirley (1965).

In the extended product basis $\{|\pm, n\rangle\}$, where n is now allowed to take negative values, the Floquet Hamiltonian has the matrix elements

$$\langle \alpha, n|H_F|\beta, m\rangle = H^{[n-m]}_{\alpha\beta} + n\omega\delta_{\alpha\beta}\delta_{nm} \tag{35}$$

where the $H^{[n]}$ come from the Fourier decomposition of the Hamiltonian H_a and α and β take the values $\pm$. This is very similar in form to H except that n can be negative and the off diagonal terms in H involve $n^{1/2}\lambda$ instead of k. However, noting that the distribution of the occupations of the photon number states is very narrow we see that the Hamiltonians have the same spectra in the physically relevant range.

In the region of interest both spectra are comprised of evenly spaced doublets. Now the doublet splitting in H gives the well known splitting of the Mollow triplet (Cohen-Tannoudji 1977), while the doublet splitting in H_F is, up to a factor of $1/\tilde{t}$, the difference in overall phases. Thus, as the two splittings are the same, the difference in overall phases is measurable as the splitting of the Mollow triplet.

4.3. Discussion

In the above we have demonstrated how the overall phases in a fermion subsystem coupled to a single boson mode can be interpreted in terms of the energy levels of the time-independent joint system. These ideas can be generalized slightly to include either more than one boson mode or anharmonicity. We discuss systems with degenerate boson modes first.

Given a constant boson–fermion Hamiltonian with N degenerate boson modes we can form the semiclassical fermion Hamiltonian by replacing the N creation operators b_j^* with the functions $e^{i\theta_j}e^{i\omega t}$. This gives the same result as if we had taken the semiclassical limit of the single-mode Hamiltonian that arises when we replace the creation operator b_j^* with the rephased creation operator $e^{i\theta_j}b^*$. For example, if we take the $E \otimes \epsilon$ Jahn–Teller Hamiltonian

$$H = EI + \omega(b_1^*b_1 + b_2^*b_2) + \lambda[-\sigma_y(b_1 + b_1^*) + \sigma_x(b_2 + b_2^*)] \tag{36}$$

discussed by Moore and Stedman (1990a), then we get the same semiclassical Hamiltonian, and so the same Berry phases, as the single mode Hamiltonian

$$H' = EI + \omega a^*a + 2\lambda \sin^{1/2}\xi\,(\sigma^+a^* + \sigma^-a) \tag{37}$$

which involves the squeezed state mode (Yuen 1976)

$$a = \mu b + \nu b^* \tag{38}$$

where $\mu = (2\sin^{1/2}\xi)^{-1}(1 + ie^{-i\xi})$, $\nu = (2\sin^{1/2}\xi)^{-1}(1 + ie^{i\xi})$ and we have replaced the two phonon creation operators b_1^* and b_2^* with b^* and $e^{i\xi}b^*$ respectively.

We can also look at boson–fermion Hamiltonians involving anharmonicity. In fact, given a general $\tilde{t}$-periodic fermionic Hamiltonian H with Fourier decomposition

$$H = \sum_{n=-\infty}^{\infty} H^{[n]}e^{in\omega t} \tag{39}$$

if we make the replacement

$$e^{in\omega t} = \begin{cases} b^{*n} & n \geq 0 \\ b^{-n} & n < 0 \end{cases} \tag{40}$$

we generate a constant boson–fermion Hamiltonian that gives H in the semiclassical limit, the Fourier components with $|n| > 1$ giving anharmonicity. Thus we can interpret the overall phases of any periodic fermion system in terms of the energy levels of a particular time-independent boson–fermion Hamiltonian.

Finally we survey other workers who have discussed Berry phase for two-level atoms. Ellinas *et al* (1989) approach this problem by adiabatically varying the detuning, coupling strength and phase properties of the laser. In effect this means that they superimpose an experimentally controllable periodic adiabatic motion on the naturally periodic motion of the system. The same kind of analysis is discussed by Andreev *et al* (1990). Breuer and Holthaus (1989) also discuss adiabatic variation of the laser parameters. They use a two-time formalism and discuss Landau–Zener transitions. Garrison and Wright (1988) and Chu *et al* (1989) analyze dissipative systems with non-self-adjoint Hamiltonians. Tewari (1989) discusses essentially the same two-level atom results as here but requires that the system be adiabatic. Finally Argawal (1988) shows how quantum beat experiments can be used to probe the topological phases associated with Rabi oscillations.

5. The forced harmonic oscillator

We now turn to a discussion of the Berry phases for forced harmonic oscillators. Consider the Hamiltonian for the periodic forced harmonic oscillator

$$H(t) = \omega b^* b + f(t)b^* + \overline{f}(t)b + \beta(t) \tag{41}$$

where f and β are $2\pi/\omega$-periodic. We will couch our discussion in the language of standard coherent states (Perelomov 1986) because of the following result which greatly simplifies the argument; it is well known that if the system starts in a standard coherent state

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} z^n (n!)^{-1/2} |n\rangle \tag{42}$$

then it will remain in a coherent state (Glauber 1966). That is, there exist solutions of the time-dependent Schrödinger equation of the form

$$|\psi(t)\rangle = e^{i\theta(t)} |z(t)\rangle. \tag{43}$$

Now it is readily shown that z satisfies the classical equation of motion

$$\dot{z} = -i(\omega z + f) \tag{44}$$

so that

$$z = e^{-i\omega t} \left(z(0) - i \int_0^t f(t') e^{i\omega t'} \, dt' \right). \tag{45}$$

Thus, as no two coherent states are equal up to a phase, in order to get cyclic evolution for a given initial coherent state $|z\rangle$ we require $z(\tilde{t}) = z(0)$ where $\tilde{t} = 2\pi/\omega$. Now from (45)

$$z(\tilde{t}) = z(0) - \mathrm{i}\int_0^{\tilde{t}} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t. \tag{46}$$

Thus we can distinguish two cases. If $\int_0^{\tilde{t}} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = 0$ then we get cyclic evolution for every initial coherent state, whereas if $\int_0^{\tilde{t}} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \neq 0$ then we do not get cyclic evolution for any initial coherent state. We note that as f is $\tilde{t}$-periodic we can expand it in a Fourier series

$$f = \sum_{n=-\infty}^{\infty} f^{[n]}\mathrm{e}^{\mathrm{i}n\omega t} \tag{47}$$

so that $\int_0^{\tilde{t}} f(t)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = \tilde{t}f^{[-1]}$ and the existence of coherent cyclic initial states depends on the non-existence of this Fourier component. In the following we analyze the case when this component vanishes and the case when it does not separately.

5.1. Case I

Here we show that if $f^{[-1]}$ vanishes then not only are all coherent states cyclic but in fact all states are cyclic. This is because the coherent states' overall phases are all the same. Thus any linear combination of them is again a cyclic initial state with the same overall phase. The result then follows from the fact that the coherent states form a complete (in fact overcomplete) set. That the coherent states must all have the same overall phase follows at once from the fact that the cyclic initial states are eigenvectors of the self-adjoint operator M in the decomposition $U = Z\mathrm{e}^{\mathrm{i}Mt}$. If any two coherent states had different overall phases then they would belong to different eigenspaces of M and so would be orthogonal, which is impossible. For completeness we will exhibit the independence of the overall phase on the initial state explicitly.

One can readily show that the phase angle θ in (43) satisfies the equation

$$-\dot{\theta} = \tfrac{1}{2}(f\bar{z} + \bar{f}z) + \beta \tag{48}$$

so that the overall phase is given by

$$-\theta(\tilde{t}) = \int_0^{\tilde{t}} \left(\tfrac{1}{2}(f\bar{z} + \bar{f}z) + \beta\right)\mathrm{d}t. \tag{49}$$

To show that this is independent of the initial state $|z(0)\rangle$ we need merely show that $\int_0^{\tilde{t}} f\bar{z}\mathrm{d}t$ is independent of $z(0)$. This follows from the equation for z and the fact that $\int_0^{\tilde{t}} f\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t = 0$ and so the overall phase is independent of the initial state as required.

We will now calculate the Berry phases for the initial coherent states. It can readily be shown that (43), (44) and (48) lead to

$$\mathrm{i}\langle\psi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -\dot{\theta} + \omega|z|^2 + \tfrac{1}{2}(\bar{z}f + z\bar{f}) \tag{50}$$

so, using the fact that the dynamical phase is given by

$$\delta = -\mathrm{i}\int_0^{\tilde{t}} \langle\psi(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle \mathrm{d}t \tag{51}$$

we find that

$$\gamma = \int_0^{\tilde{t}} \left[\omega|z|^2 + \tfrac{1}{2}(\bar{z}f + z\bar{f})\right] \mathrm{d}t \tag{52}$$

which can readily be converted into an integral involving only f and β if required.

5.2. Case II

Here we show that if $f^{[-1]}$ does not vanish then not only are no coherent states cyclic, but no states are cyclic. This provides an example of a system with an infinite-dimensional Hilbert space whose evolution operator at time $\tilde{t}$ does not have any eigenvectors. This is in marked contrast to the case of systems with finite-dimensional Hilbert spaces, as there the evolution operator is always diagonalizable and there is always a complete set of cyclic initial states.

We use the fact that the set of coherent states is complete, i.e. there exists a spectral resolution of unity

$$\int \mathrm{d}\mu(z)|z\rangle\langle z| = I \tag{53}$$

where, for $z = z_1 + \mathrm{i}z_2$ with z_1 and z_2 real, the measure is given by $\mathrm{d}\mu(z) = \pi^{-1}\mathrm{d}z_1\mathrm{d}z_2$. Thus an arbitrary state can be written

$$\psi = \int \mathrm{d}\mu(z)b(z)|z\rangle \tag{54}$$

where $b(z) = \langle z|\psi\rangle$. Further if $\psi' = \int \mathrm{d}\mu(z)b'(z)|z\rangle$ then $\psi = \psi'$ if and only if $b(z) = b'(z)$ for all z. That this is so is most easily seen by considering the magnitude of $\psi - \psi'$.

Now imagine that $\psi(0) = \int \mathrm{d}\mu(z)b(z)|z\rangle$ is a cyclic initial state. Then

$$\psi(\tilde{t}) = \int \mathrm{d}\mu(z)\mathrm{e}^{\mathrm{i}\chi}b(z)|z\rangle \tag{55}$$

where χ is the overall phase. Now putting $k = \tilde{t}f^{[-1]}$ and noting that $z(\tilde{t}) = z(0) - \mathrm{i}k$ we can see that $|z\rangle$ evolves into $\mathrm{e}^{\mathrm{i}\chi(z)}|z - \mathrm{i}k\rangle$. Thus

$$\psi(\tilde{t}) = \int \mathrm{d}\mu(z)b(z + \mathrm{i}k)\mathrm{e}^{\mathrm{i}\chi(z+\mathrm{i}k)}|z\rangle \tag{56}$$

where we have used the fact that $\mathrm{d}\mu(z + \mathrm{i}k) = \mathrm{d}\mu(z)$ for constant k. Hence comparing (55) and (56) we must have

$$\mathrm{e}^{\mathrm{i}\chi}b(z) = \mathrm{e}^{\mathrm{i}\chi(z+\mathrm{i}k)}b(z + \mathrm{i}k) \tag{57}$$

which gives

$$|b(z)|^2 = |b(z + \mathrm{i}nk)|^2 \tag{58}$$

for all integral n.

We now show that this relation means that $\psi(0)$ cannot be normalized so that there are no cyclic initial states. For convenience we take k to be real, however the proof is equivalent for arbitrary k. For $\psi(0)$ to be normalized we require

$$\int \mathrm{d}\mu(z)|b(z)|^2 = 1. \tag{59}$$

We write $\int \mathrm{d}\mu(z)|b(z)|^2 = \pi^{-1}\int_{-\infty}^{\infty} V(z_1)\mathrm{d}z_1$, where $V(z_1) = \int_{-\infty}^{\infty}|b(z_1 + iz_2)|^2\mathrm{d}z_2$. Now we can split the integral over z_2 into the sum

$$V(z_1) = \sum_{n=-\infty}^{\infty}\int_{nk}^{(n+1)k}|b(z_1 + \mathrm{i}z_2)|^2\mathrm{d}z_2. \tag{60}$$

Using the periodicity of $|b(z)|^2$ under translation by k along the imaginary axis we can recast (60) into the form

$$V(z_1) = \sum_{n=-\infty}^{\infty}\int_{0}^{k}|b(z_1 + \mathrm{i}z_2)|^2\mathrm{d}z_2 \tag{61}$$

which is obviously infinite. Thus there are no normalizable cyclic initial states for the periodic forced harmonic oscillator with $f^{[-1]}$ non-zero.

5.3. Discussion

In the above we have shown how the periodic forced harmonic oscillator may or may not have any cyclic initial states depending on the existence of a certain Fourier component in the forcing term. This work can be generalized slightly by noting that Glauber (1966) showed that the most general form of periodic oscillator Hamiltonian that preserves coherence is in fact

$$H(t) = \omega(t)b^*b + f(t)b + \overline{f}(t)b + \beta(t) \tag{62}$$

where we take ω, f and β to be $\tilde{t}$-periodic for some $\tilde{t}$. This leads to the same equation for z as for constant ω and we find that

$$z = \left[z(0) - \mathrm{i}\int_0^{\tilde{t}} f\exp\left(\mathrm{i}\int_0^{t'}\omega\mathrm{d}t''\right)\mathrm{d}t'\right]\exp\left(-\mathrm{i}\int_0^{\tilde{t}}\omega\mathrm{d}t\right) \tag{63}$$

so that we get cyclic evolution for the unique initial coherent state with

$$z(0) = \frac{-\mathrm{i}\exp\left(-\mathrm{i}\int_0^{\tilde{t}}\mathrm{d}t\right)}{\left[1 - \exp\left(-\mathrm{i}\int_0^{\tilde{t}}\mathrm{d}t\right)\right]}\int_0^{\tilde{t}} f\exp\left(\mathrm{i}\int_0^{t}\omega\mathrm{d}t'\right)\mathrm{d}t \tag{64}$$

where we have assumed that $\exp(-\mathrm{i}\int_0^{\tilde{t}}\mathrm{d}t) \neq 1$.

Finally we survey other workers who have discussed similar problems. Chaturvedi *et al* (1987) discuss Berry phase for the forced harmonic oscillator by adiabatically varying the forcing term $f(t)$. In effect this means that they choose f to have a much longer period than the period $2\pi/\omega$ used here. Brihaye *et al* (1990) discuss Berry phases for coherent states in a general context.

6. Conclusion

In the above we have discussed the relationship between Berry phase and the time dependence of the Hamiltonian and provided two examples where the time dependence of the Hamiltonian is generated in different ways. There is much scope for expanding the ideas of this work. One obvious one we are currently studying is how to use the two-boson mode formalism to suppress the usually dominant rotating wave part of the electron–photon interaction, either by making an appropriate choice of phase difference between two laser beams with different polarizations incident upon the same two-level atom or by using elliptically polarized light.

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